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SOLUTION OF THE DIFFERENTIAL EQUATIONS OF HEAT AND MASS TRANSFER FOR LAMINAR FLOW OF A BINARY GAS MIXTURE OVER A FLAT PLATE

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An attempt is made to elucidate the increased mass transfer resulting from flow pulsations on the basis of a solution of the differential equations of heat and mass transfer.

When a binary gas mixture flows over a moist capillary-porous plate the heat transfer is due to vapor diffusion. If we denote mass flow normal to the plate by $j = \rho w_y = \mathrm{const}$, then

$$\frac{\partial^2 \theta}{\partial y^2} - \frac{w_y \partial \theta}{a \partial y} - \frac{w_x \partial \theta}{a \partial x} = 0. \tag{1}$$

This equation may be solved by an operational method (with boundary conditions: $y = 0,\ \theta = 0,\ y \to \infty,\ \theta = t_m - t_w = \theta_c\ \ x = 0,\ \ \theta = \theta_m$):

$$\frac{\theta(x,\ y)}{\theta_m} = 1 - \frac{1}{2}\left[\operatorname{erfc}\left(\frac{\sqrt{w_x}\, y}{2\sqrt{ax}} - \frac{w_y}{2a}\sqrt{\frac{ax}{w_x}}\right) + \right.$$
$$\left. + \exp\frac{w_y}{a} y \operatorname{erfc}\left(\frac{\sqrt{w_x}\, y}{2\sqrt{ax}} + \frac{w_y}{2a}\sqrt{\frac{ax}{w_x}}\right)\right]. \tag{2}$$

In these conditions the local Nu number is

$$\mathrm{Nu}_x = \frac{\alpha_x x}{\lambda} = x \frac{\partial}{\partial y}\left[\frac{\theta(x,\ 0)}{\theta_m}\right]. \tag{3}$$

Differentiating (2) with respect to y and putting y = 0, we have

$$\mathrm{Nu}_x = \frac{1}{\sqrt{\pi}}\sqrt{\frac{w_x x}{a}} \exp\left(-\frac{w_y^2}{4a^2}\frac{ax}{w_x}\right) -$$
$$- \frac{w_y x}{2a} \operatorname{erfc}\frac{w_y}{2a}\sqrt{\frac{ax}{w_x}}. \tag{4}$$

The local value of the Pe number, referred to the mean integral velocity w_x, will be $\mathrm{Pe}_x = w_x x/a$, and referred to the mean integral velocity $w_y - \mathrm{Pe}_x^* = w_y x/a$.

Taking this into account, we obtain the local Nu number in the form:

$$\mathrm{Nu}_x = \frac{1}{\sqrt{\pi}}\sqrt{\mathrm{Pe}_x} \exp\left(-\frac{\Pi^2}{4}\right) - \frac{1}{2}\Pi\sqrt{\mathrm{Pe}_x} \operatorname{erfc}\frac{\Pi}{2} =$$
$$= \frac{1}{\sqrt{\pi}}\sqrt{\mathrm{Pe}_x}\left[\exp\left(-\frac{\Pi^2}{4}\right) - \frac{\sqrt{\pi}}{2}\Pi \operatorname{erfc}\frac{\Pi}{2}\right], \tag{5}$$

where $\Pi = \mathrm{Pe}_x^*/\sqrt{\mathrm{Pe}_x}$.

Thus, the local Nu number is expressed as a function of the local Pe number with account for the transverse mass flow.

When evaporative porous cooling takes place, the evaporating surface sinks a certain distance ζ into the porous body. The differential equation of heat transfer retains its previous form:

$$w_x \frac{\partial t}{\partial x} + w_y \frac{\partial t}{\partial y} = a \frac{\partial^2 t}{\partial y^2}. \tag{6}$$

The boundary conditions are:

$$t(0,\ y)=t_m,\ t(x,\ \infty)=t_m,\ t(x,\ \zeta)=t_w,$$

$$-\lambda\frac{\partial t(0,\ x)}{\partial y} = -\lambda_T\frac{\partial t(0,\ x)}{\partial y}=\frac{\lambda_T}{\zeta}\,[t(0,\ x)-t_w],$$

or

$$-\frac{\partial t(0,\ x)}{\partial y}+H\,[t(0,\ x)-t_w]=0. \tag{6'}$$

It should be noted that the temperature at the evaporation surface is the wet-bulb temperature (adiabatic conditions).

After changing variables and solving, we obtain

$$T(y,\ s)=\frac{\theta_m}{s}-\frac{\theta_m}{s}\left[1+\frac{1}{H}\left(\sqrt{\frac{w_y^2}{4a^2}+s\frac{w_x}{a}}-\frac{w_y}{2a}\right)\right]^{-1}\times$$
$$\times\exp\left(-\sqrt{\frac{w_y^2}{4a^2}+s\frac{w_x}{a}}+\frac{w_y}{2a}\right)y. \tag{7}$$

The local Nu number for the case of porous cooling with a sunken evaporating surface is

$$\mathrm{Nu}_x=\frac{x}{\theta_m}\,\frac{\partial t(x,\ \zeta)}{\partial y} \tag{8}$$

or

$$\frac{\mathrm{Nu}_x}{x}=\frac{1}{\theta_m}\,T'(x,\ \zeta).$$

For the plate surface (ζ = 0) the inverse transform will be

$$L^{-1}\left[\frac{\mathrm{Nu}_x}{x}\right]=\frac{1}{\theta_m}\,T'(x,\ 0)=\frac{1}{s}\left(\sqrt{\frac{w_y^2}{4a^2}+s\frac{w_x}{a}}-\frac{w_y}{2a}\right)\times$$
$$\times\left[1+\frac{1}{H}\left(\sqrt{\frac{w_y^2}{4a^2}+s\frac{w_x}{a}}-\frac{w_y}{2a}\right)\right]^{-1}= \tag{9}$$
$$=\frac{H}{s}-H\left\{s\left[1+\frac{1}{H}\left(\sqrt{\frac{w_y^2}{4a^2}+s\frac{w_x}{a}}-\frac{w_y}{2a}\right)\right]\right\}^{-1}.$$

Let us examine the expression in parentheses in more detail:

$$\left(\sqrt{\frac{w_y^2}{4a^2}+s\frac{w_x}{a}}-\frac{w_y}{2a}\right)=\sqrt{\frac{w_x}{a}}\sqrt{\frac{w_y^2}{4aw_x}+s}-$$
$$-\frac{w_y}{2a}\cong\sqrt{\frac{w_x s}{a}}-\frac{w_y}{2a},$$

since in our case the velocity w_y is negligibly small in comparison with w_x.

Consequently,

$$\frac{\mathrm{Nu}_x}{x}=L^{-1}\left\{\frac{H}{s}-H\left[\frac{1}{H}\sqrt{\frac{w_x}{a}}\,s\left(\sqrt{s}+\frac{1-w_y/2Ha}{\sqrt{w_x/a}/H}\right)\right]^{-1}\right\}. \tag{10}$$

We find the original of the function from the transform [1]:

$$\mathrm{Nu}_x=Hx-Hx\left(1-\frac{1}{H}\,\frac{w_y}{2a}\right)^{-1}+$$
$$+Hx\left(1-\frac{1}{H}\,\frac{w_y}{2a}\right)^{-1}\exp\left(1-\frac{w_y}{2Ha}\right)^2x\left(\frac{1}{H}\sqrt{\frac{w_x}{a}}\right)^{-2}\times \tag{11}$$

$$\times \operatorname{erfc}\left(1-\frac{1}{H}\frac{w_y}{2a}\right)\times\sqrt{x}\left(\frac{1}{H}\sqrt{\frac{w_x}{a}}\right)^{-1}. \qquad (11) \text{ (cont'd)}$$

If we neglect heat transfer due to vapor diffusion, i.e., if $w_y = 0$, Eq. (11) takes the form:

$$\mathrm{Nu}_x = Hx\exp H^2x\frac{a}{w_x}\operatorname{erfc} H\sqrt{ax/w_x}. \qquad (12)$$

We put $K = Hx/\sqrt{\mathrm{Pe}_x}$, then $\mathrm{Nu}_x = \sqrt{\mathrm{Pe}_x}K\exp K^2\operatorname{erfc}K$ [2], or

$$\mathrm{Nu}_x\sqrt{\pi}/\sqrt{\mathrm{Pe}_x} = \sqrt{\pi}K\exp K^2\operatorname{erfc}K. \qquad (13)$$

Returning to (11) and making the substitution:

$$w_yx/a = \mathrm{Pe}_x^*,\ \mathrm{Pe}_x^*/\sqrt{\mathrm{Pe}_x} = \Pi \text{ and } \mathrm{Pe}_x^*/Hx = \Pi/K,$$

we have

$$\mathrm{Nu}_x = Hx\left\{1-\frac{1}{1-\Pi/2K}\left[1-\exp\left(K^2\left(1-\frac{\Pi}{2K}\right)^2\right)\times\right.\right.$$
$$\left.\left.\times\operatorname{erfc}K\left(1-\frac{\Pi}{2K}\right)\right]\right\}. \qquad (14)$$

If $K \to \infty$ in (13), i.e., if evaporation takes place at the surface of the plate, we obtain $\mathrm{Nu}_x\sqrt{\pi}/\sqrt{\mathrm{Pe}_x} = 1$. Denoting the left side of this equation by N, we substitute in it the value of Nu_x from (14):

$$N = \frac{K\sqrt{\pi}}{1-\Pi/2K}\left\{\exp\left[K^2\left(1-\frac{\Pi}{2K}\right)^2\right]\times\right.$$
$$\left.\times\operatorname{erfc}\left(1-\frac{\Pi}{2K}\right)K-\frac{\Pi}{2K}\right\}. \qquad (15)$$

We thus have an expression for the relative increase in heat transfer during evaporation of moisture from a porous body with account for the depression of the evaporating surface in the presence of a transverse mass flux.

When the porous body is wetted continuously (water seeping through the pores), i.e., when the drying process occurs in a period of constant velocity, the temperature of the evaporating surface is constant and the value of ζ is very small. In practice, therefore, it is impossible to measure accurately the surface temperature of the body, because, in practice, thermocouples attached to the surface of the plate will show the wet-bulb temperature.

Let us determine the temperature distribution on the surface of the plate along the x axis. For this purpose we must return to (7), written in the form

$$T(y,s) = \frac{\theta_m}{s}-\frac{1}{s}\,\frac{\theta_m\exp\left(\frac{w_yy}{2a}\right)}{1/H}\times$$
$$\times\frac{\exp\left(-\sqrt{w_y^2/4a^2+s\,w_x/a}\right)y}{H-w_y/2a+\sqrt{w_y^2/4a^2+s\,w_x/a}}. \qquad (16)$$

We find the original of the function from the given transform [1]:

$$\frac{\theta(x,y)}{\theta_m} = \frac{H-w_y/2a}{H-w_y/a}\exp\left[\left(H^2-\frac{Hw_y}{a}\right)\frac{xa}{w_x}+Hy\right]\times$$
$$\times\operatorname{erfc}\left[\left(H-\frac{w_y}{2a}\right)\sqrt{\frac{ax}{w_x}}+\frac{y}{2}\sqrt{\frac{w_x}{ax}}\right]+ \qquad (17)$$
$$+\frac{1}{2}\operatorname{erfc}\left(\frac{w_y}{2a}\sqrt{\frac{ax}{w_x}}-\frac{y}{2}\sqrt{\frac{w_x}{ax}}\right)-$$

$$-\frac{H \exp w_y y/a}{2(H - w_y/a)} \operatorname{erfc}\left(\frac{w_y}{2a}\sqrt{\frac{ax}{w_x}} + \frac{y}{2}\sqrt{\frac{w_x}{ax}}\right). \tag{17 (cont'd)}$$

Therefore the temperature difference $t(x, y) - t_w$ is related to the psychrometric difference $t_m - t_w$ as follows:

$$\frac{t(x, y) - t_w}{t_m - t_w} = \frac{H - w_y/2a}{H - w_y/a} \exp\left[\left(H^2 - \frac{Hw_y}{a}\right)\frac{ax}{w_x} + Hy\right] \times$$

$$\times \operatorname{erfc}\left[\left(H - \frac{w_y}{2a}\right)\sqrt{\frac{ax}{w_x}} + \frac{y}{2}\sqrt{\frac{w_x}{ax}}\right] + \frac{1}{2}\operatorname{erfc}\left(\frac{w_y}{2a}\sqrt{\frac{ax}{w_x}} - \frac{y}{2}\sqrt{\frac{w_x}{ax}}\right) - \tag{18}$$

$$-\frac{H \exp w_y y/a}{2(H - w_y/a)} \operatorname{erfc}\left(\frac{w_y}{2a}\sqrt{\frac{ax}{w_x}} + \frac{y}{2}\sqrt{\frac{w_x}{ax}}\right).$$

The temperature at the surface, i.e., at y = 0, is

$$\frac{t(x, 0) - t_M}{t_c - t_M} = \frac{H - w_y/2a}{H - w_y/a} \exp\left[\left(H^2 - \frac{Hw_y}{a}\right)\frac{ax}{w_x}\right] \times$$

$$\times \operatorname{erfc}\left[\left(H - \frac{w_y}{2a}\right)\sqrt{\frac{ax}{w_x}}\right] +$$

$$+\frac{1}{2}\operatorname{erfc}\frac{w_y}{2a}\sqrt{\frac{ax}{w_x}} - \frac{H}{2(H - w_y/a)}\operatorname{erfc}\frac{w_y}{2a}\sqrt{\frac{ax}{w_x}} = \tag{19}$$

$$= \frac{H - w_y/2a}{H - w_y/a} \exp\left[\left(H^2 - \frac{Hw_y}{a}\right)\frac{ax}{w_x}\right] \times$$

$$\times \operatorname{erfc}\left(H - \frac{w_y}{a}\right)\sqrt{\frac{ax}{w_x}} - \frac{w_y}{2(H - w_y/a)a}\operatorname{erfc}\frac{w_y}{2a}\sqrt{\frac{ax}{w_x}}.$$

In this case the heat transfer coefficient is calculated as the ratio of heat flux to the psychrometric difference $\alpha_{x_M} = q/(t_m - t_w)$, and the local Nu number is expressed as

$$\mathrm{Nu}_{x_M} = \frac{\alpha_{x_M} x}{\lambda} = \frac{x}{t_m - t_w}\frac{\partial t(x, 0)}{\partial y}. \tag{20}$$

To find Nu_{x_M} it is necessary to differentiate (18) with respect to y and equate y to zero. After substituting and simplifying, we obtain

$$\mathrm{Nu}_{x_M} = \frac{1 - \Pi/2K}{1 - \Pi/K}\left[Hx \exp(K^2 - \Pi K) \operatorname{erfc}\left(K - \frac{\Pi}{2}\right)\right] - $$

$$-\frac{1}{2}\frac{\mathrm{Pe}_x^*}{1 - \Pi/K}\operatorname{erfc}\frac{\Pi}{2}. \tag{21}$$

Then the coefficient N_m, showing the relative change in Nu_{x_M}, and hence in the heat transfer coefficient α_{x_M}, due to the evaporating surface sinking into the body, is expressed as

$$N_M = \frac{\sqrt{\pi}}{1 - \Pi/K}\left\{\left[1 - \frac{\Pi}{2K}\right] K \exp(K^2 - \Pi K) \times \operatorname{erfc}\left(K - \frac{\Pi}{2}\right) - \frac{\Pi}{2}\operatorname{erfc}\frac{\Pi}{2}\right\}. \tag{22}$$

It should be noted that the value of Π may be assumed constant for given conditions of flow of gas mixture over the plate. Then, taking into account that $K = \lambda_r x/\zeta\lambda\sqrt{\mathrm{Re}_x}$, we may construct the curve $N_M = f(\zeta)$ for various ζ (table).

The differential equation of mass transfer has the form

$$w_x \frac{\partial \rho_{10}}{\partial x} + w_y \frac{\partial \rho_{10}}{\partial y} = a_m \frac{\partial^2 \rho_{10}}{\partial x^2}. \tag{23}$$

We assume the boundary conditions to be

$$y \to \infty, \quad \rho_{10} = \rho_{10_m},$$

$$x = 0, \quad \rho_{10}(0, y) = \rho_{10_m},$$

$$y = 0, \quad -a_m \rho \frac{\partial \rho_{10}}{\partial y} = -a_{m_T} \rho \frac{\partial \rho_{10}}{\partial y} =$$

$$= \frac{a_{m_T}}{\zeta} \rho \left[\rho_{10}(0, x) - \rho_{10_W}\right]. \tag{24}$$

$$\rho_1 = p_1 \frac{M_1}{RT}; \quad \rho = p \frac{M}{RT}; \quad \rho_{10} = \frac{\rho_1}{\rho} = \frac{M_1 p_1}{Mp};$$

$$\rho_{10_m} = \frac{M_1 p_{cp}}{Mp}; \quad \rho_{10_W} = \frac{M_1 p_{HM}}{Mp}.$$

We may write (24) as

$$-\frac{\partial \rho_{10}(0, x)}{\partial y} + H_m [\rho_{10_M} - \rho_{10}(0, x)] = 0. \tag{25}$$

Examining (23) and (25), we find that when the notation is changed they coincide with (1) and (6'), and their solution will be analogous to that of (18) and (22):

$$\frac{\rho_{10_W} - \rho_{10}(x, y)}{\rho_{10_W} - \rho_{10_m}} = \frac{H_m - w_y/2a_m}{H_m - w_y/a_m} \times$$

$$\times \exp\left[\left(H_m^2 - \frac{H_m w_y}{a_m}\right)\frac{x a_m}{w_x} + H_m y\right] \times$$

$$\times \operatorname{erfc}\left[\left(H_m - \frac{w_y}{2a_m}\right)\sqrt{\frac{x a_m}{w_x}} + \frac{y}{2}\sqrt{\frac{w_x}{a_m x}}\right] +$$

$$+ \frac{1}{2}\operatorname{erfc}\left(\frac{w_y}{2a_m}\sqrt{\frac{x a_m}{w_x}} - \frac{y}{2}\sqrt{\frac{w_x}{a_m x}}\right) - \tag{26}$$

$$- \frac{H_m \exp w_y y/a_m}{2(H_m - w_y/a_m)} \operatorname{erfc}\left(\frac{w_y}{2a_m}\sqrt{\frac{x a_m}{w_x}} + \frac{y}{2}\sqrt{\frac{w_x}{x a_m}}\right);$$

$$N'_M = \frac{\sqrt{\pi}\,\mathrm{Nu}'_{x_M}}{\sqrt{\mathrm{Pe}'_x}} = \frac{\sqrt{\pi}}{1 - \Pi_m/K_m}\left[\left(1 - \frac{\Pi_m}{2K_m}\right) K_m \exp(K_m^2 - \Pi_m K_m) \times\right.$$

$$\left.\times \operatorname{erfc}\left(K_m - \frac{\Pi_m}{2}\right) - \frac{\Pi_m}{2}\operatorname{erfc}\frac{\Pi_m}{2}\right].$$

Analyzing (22) and (26), we find that they are similar, and so N'_m also decreases with increasing ζ.

Changes in Heat and Mass Transfer When the Evaporating Surface Sinks into the Body

Mass transfer		Heat transfer		Mass transfer		Heat transfer	
$\zeta \cdot 10^3$, m	N_M	$\zeta \cdot 10^3$, m	N'_M	$\zeta \cdot 10^3$, m	N_M	$\zeta \cdot 10^3$, m	N'_M
143.9889	0.15876	5.4203	0.158859	7.9993	0.88760	0.3011	0.888024
71.9943	0.28678	2.7101	0.286513	7.1994	0.90429	0.2710	0.904560
47.9963	0.39050	1.8067	0.390324	6.5449	0.91767	0.2464	0.917109
35.9972	0.47580	1.3551	0.472725	5.9995	0.92855	0.2258	0.939165
28.7977	0.54542	1.0841	0.545284	5.5388	0.93186	0.2085	0.937154
23.9981	0.60356	0.9034	0.603522	5.1424	0.93884	0.1936	0.944385
20.5698	0.65216	0.7743	0.652145	4.7996	0.94524	0.1807	0.951105
17.9986	0.69313	0.6775	0.693091	4.4996	0.94537	0.1694	0.951581
15.9987	0.72778	0.6022	0.727511	4.2349	0.94723	0.1594	0.952632
14.3989	0.75744	0.5420	0.757368	3.9996	0.94750	0.1505	0.953502
11.9991	0.80403	0.4517	0.806370	3.7892	0.94768	0.1426	0.954335
10.2849	0.83980	0.3872	0.839864	3.5997	0.94784	0.1355	0.956067
8.9993	0.86707	0.3387	0.866881	0.00	0.99826	0.00	0.999740

It is known from experiment that the depth to which the evaporating surface sinks depends on the air flow conditions over the body. For equal values of the Re number in stable and pulsating flow, this depth ζ varies. As heat and mass transfer intensify due to pulsations, ζ decreases, but N_M and N'_M increase, although to different degrees. Thus, for heat transfer, when ζ decreases, N_M increases on that part of the curve $N_M = f(\zeta)$ where the variation of N_M is slight and the value of N_M is close to its limit. For mass transfer, for the same variation of the depth ζ, the value of N'_M increases considerably. Hence it is clear that as ζ decreases, heat and mass transfer increase differently.

We thus reach the conclusion that pulsations of the air flowing over a moist porous body reduce the depth ζ of the evaporating surface and considerably increase only the mass transfer, as has been verified experimentally in [3, 4].

NOTATION

t_m — temperature of the vapor-air medium; t_w — wet-bulb temperature; $H = \lambda_t/\lambda\zeta$ — a quantity similar to the relative heat transfer coefficient; $H_m = a_{mt}/a_m\zeta$ — a quantity similar to the relative mass transfer coefficient; N_M — a dimensionless quantity describing the relative increase in local Nusselt number when moisture is evaporated from a porous body as compared with evaporation at the surface of the body.

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